MATHEMATICAL MODEL FOR CHESSBOARD WAVES

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Abstract

Chessboard waves are often seen near the island of Rhe (*Isle de Rhe*) in France. The pattern is very striking because of its strange, regular, geometrical form. It is believed to be a cross-sea, and to occur when waves from different directions meet each other at close to right angles. The group considered possible mechanisms for the formation of the waves and concluded that it is most likely that they are due to the interaction of solitary waves generated by bottom topography where flows from different directions meet. Some simple example computations are performed to demonstrate that this is one possible mechanism.

1 The problem

Chessboard waves are often seen near the island of Rhe (*Isle de Rhe*) in France. The pattern is believed to be a cross-sea, and to occur when waves from different directions meet each other at close to right angles. Chessboard waves are dangerous for large ships, boats, swimmers and surfers. The study group was asked to consider the mechanisms that may be the cause of these wave patterns and to derive a mathematical model for them. Images and a video of these waves can be seen at here

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2 Early considerations

The group began by considering the local conditions to see if there was some phenomenon that might be the cause, or some unique factor that would create the unusual wave conditions. Using coastal maps we found that the directions of the wind and the current seem to be perpendicular. Topographic maps available on the internet did not include sufficiently small contour intervals to see the local bottom shape. However, it appears that there are oyster farms in the region which suggests a reasonably flat bottom.

Waves fall into several categories that need to be considered. In general waves seen on the ocean are not particularly large and are termed sinusoidal waves, characterised by regular up and down motions. Sinusoidal waves are those that you see over the side of a boat in a river or the ocean. Waves approaching the beach are no longer of this type and are called nonlinear waves. Waves that have larger amplitude develop from these sinusoidal waves and these have narrow crests and broader troughs. Tsunami are waves with extremely long wavelength and are usually generated by some tectonic activity. Another class of waves is those termed solitary waves that consist of a single wave that propagates at a fixed height and travels for long distances.

3 Ocean waves

Historical consideration of waves on the ocean began with the analysis of the linearized equations of fluid dynamics [1, 11, 12]. In the problem being considered in this study group, it would seem likely that the wave field produced is generated by the interactions between wind waves and bottom topography, perhaps acting in different directions. Therefore we begin by a simple recapitulation of the basic mathematical equations for water waves in water of finite depth. Assuming the fluid motion is described by the irrotational flow of an inviscid, incompressible fluid with a surface given by $z = \eta(x, y, t)$ and conditions on the surface given by a kinematic condition (fluid particles on the surface remain on the surface) and a Bernoulli condition (the pressure of the air above the waves is uniform and constant), we can define a velocity potential $\phi(x, y, z, t)$ such that the velocity $\mathbf{v} = \nabla\phi$. The model is therefore

$$\nabla^2\phi = 0, \qquad -\infty < z < \eta(x, t), \qquad -\infty < x < \infty, t > 0 \ , \tag{1}$$

subject to the dynamic (Bernoulli) condition that the pressure on the water surface must be atmospheric, i.e.

$$\phi_t = -\frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) - g\eta \quad \text{on} \quad z = \eta(x, y, t) \ , \tag{2}$$

where g is the gravity constant, and the kinematic condition given by

$$\eta_t + \eta_x\phi_x + \eta_y\phi_y - \phi_z = 0 \quad \text{on} \quad z = \eta(x, y, t). \tag{3}$$

Finally there is a condition that there is no flow through the bottom of the ocean, so that

$$B_x\phi_x + B_y\phi_y - \phi_z = 0 \quad \text{on} \quad z = B(x, y) \ , \tag{4}$$

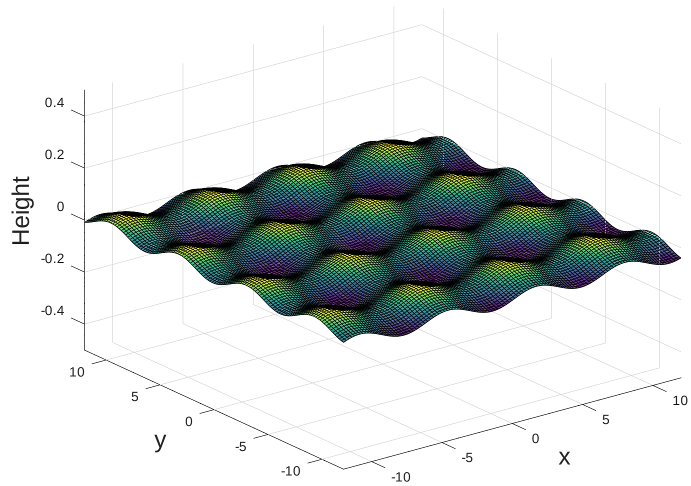

Figure 1: Pattern of intersecting waves assuming sinusoidal, or normal wind-driven waves. This looks nothing like the chessboard waves seen at Isle de Rhe.

where $B(x, y)$ defines the shape of the bottom. Solutions for the full equations are obtained for flow over various obstacles in the work of Forbes and Schwartz [5], Zhang and Zhu [13] and Holmes et al [9]. At smaller values of stream flow, a wave train develops downstream as the flow goes over the obstacle. As the flow rate increases or the obstacles get higher, the waves take on a nonlinear appearance with narrow crests and broad troughs. However, there are some unique flows in which the bottom topography results in a cancellation of all downstream waves [6, 9]. A critical parameter in this work is the Froude number, which is defined as

$$Fr = \frac{U}{\sqrt{gH}}. \tag{5}$$

If the Froude number is small it means the flow is relatively slow, while if Fr is larger than unity, $Fr > 1$, then linear theory precludes the existence of waves.

3.1 Linear waves

In some rare circumstances ocean waves may have very large amplitude relative to their length, but in general wind generated ocean waves are of “linear” type. That is to say the disturbance to the water surface is small.

In that case, the product terms can be omitted from the equations and the problem solved by expanding about the undisturbed level of the free surface. Assuming the flow does not vary perpendicular to the direction of the waves, the problem becomes two dimensional.

With these simplifications the surface conditions (2) and (3) reduce to

$$\phi_t = -g\eta \quad \text{with} \quad \eta_t = \phi_z \quad \text{on} \quad z = 0$$

and therefore

$$\phi_{tt} + g\phi_z = 0 \quad \text{on} \quad z = 0. \tag{6}$$

If the bottom $z = B(x) = -h$ is flat, then we require $\phi_z(x, -h, t) = 0$ on $z = -h$, and we can assume a solution of the form

$$\phi(x, z, t) = Ux + \int_0^\infty C(k, t) \cosh(k(z + h)) \cos(kx)\; dk. \tag{7}$$

This choice satisfies Laplace's equation (1) and the bottom condition (4). Substituting (7) into the remaining surface condition (6) gives

$$\int_0^\infty [C_{tt} \cosh(k(z + h)) + gk \sinh(k(z + h))C] \cos kx\; dk = 0, \tag{8}$$

requiring

$$C_{tt} \cosh(k(z + h)) + gk \sinh(k(z + h))C = 0\ ,$$

which implies that

$$C(k, t) = D(k) \cos \omega t + E(k) \sin \omega t$$

where

$$\omega = \sqrt{gk \tanh kh}$$

is known as the dispersion relation and relates the wave frequency to the water depth and wavenumber. The unknown quantities, $D(k), E(k)$ can be determined from the initial conditions, and the extension to the case with bottom topography can be found in Lamb [11] or more recently in Holmes et al [9]. Using (6) we note that the surface shape can be written in the general form

$$\eta(x, t) = -\frac{1}{g} \int_0^\infty G(k)\omega \cosh(kh) \sin(kx - \omega t)dk \tag{9}$$

which gives waves travelling to the right with speed

$$c = \omega/k = \sqrt{g/k \tanh kh}$$

and $G(k)$ is some function determined by the initial conditions. The important thing is that this has identified the characteristic behaviour of the waves and thus can be used to consider how they would look if they were in some kind of crossing pattern.

Since the equations for this situation are linear, they can be added together in any configuration we wish. Figure 1 shows the pattern that would be seen if linear, sinusoidal waves were to cross perpendicular to each other. Clearly this pattern is not what is seen at the location under consideration as they are smoothly and continously oscillating as they interact. On the Isle de Rhe there are long flat regions between isolated waves that propogate perpendicular to each other.

3.2 Solitary waves - weakly nonlinear equations

Solitary waves appear to have been first described by Russell, as noted by Boussinesq [2], and subsequently there has been much research on the subject. These waves are found as solutions to equations that have names such as Schrödinger's equation (see for example [3], 10]), as considered in the 2019 South African Study Group [4] and the Korteweg-de Vries (KdV) equation [10], for example as described in Grimshaw [7]. Unfortunately, these equations are nonlinear and derived based on waves travelling in a single direction and not at an angle to each other. It was as a consequence not possible within the week to provide a mathematical model to describe the full situation of solitary waves crossing relative to different types of meteorological and bottom forcing.

The KdV equation is appropriate for water of finite depth, and it occurs in cases where the Froude number, Fr, is close to one. An excellent summary is given in the paper of Grimshaw [8]. Starting with equations (1-4) and making the assumption that the wavelength is much longer than their amplitude A, that is, $\lambda/A << 1$, one arrives at the KdV equation [10].

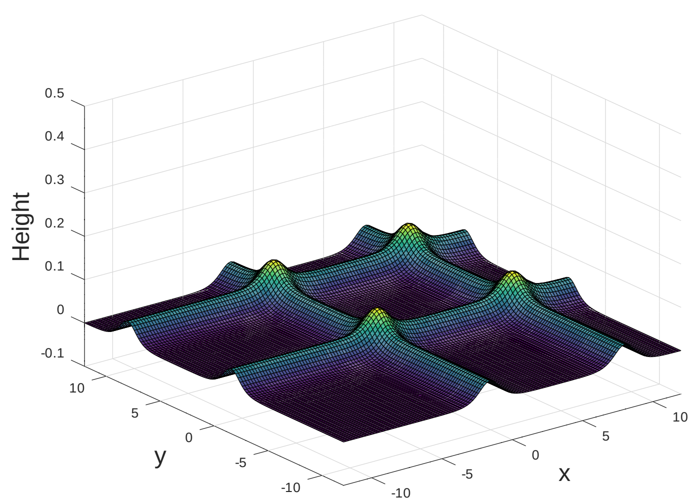

Figure 2: Pattern of intersecting waves assuming solitary waves of KdV form. Although this is **not** a formal solution to the full two-dimensional problem, it does bear a strong resemblance to the waves on Isle de Rhe.

The KdV equation has a classical solution in the form of a sech function. An illustration of two separate sets of solitary waves perpendicular to each other is shown in Figure 2, and this has a great similarity to the waves seen at Isle de Rhe. It is important to note that combining the solutions in this way is **not** mathematically valid because the KdV equation is not linear, and so the solutions can not be added. However, when travelling in the same direction it is known that solitary waves pass through each other without loss of their form [8] and so together with the long, flat undisturbed region between the waves, this is not a completely unrealistic picture of how such a pattern would look. In light of this picture the group decided that it is likely that these are the appropriate waves to be considered in the mathematical model, but in the time available it was not possible to derive and solve a full model for solitary waves crossing over bottom topography, and so instead we considered whether some bottom topography could generate solitary waves. If so, then it is not unreasonable to assume that the cause of the wave pattern is the interaction of sea swell from different directions encountering some unique bottom topography, as suggested by Figure 2.

Grimshaw et al [7, 8] considered what is known as the modified or forced KdV equation (fKdV) for flow of a fluid of depth h with bottom topography of the form $\hat{F}(\hat{x})$ and flow speed U. The appropriate form is

$$-\hat{A}_{\hat{t}} - \hat{\Delta}\hat{A}_x - \mu\hat{A}\hat{A}_x + \lambda\hat{A}_{\hat{x}\hat{x}\hat{x}} + \frac{c}{2}\hat{F}_{\hat{x}} = 0 \tag{10}$$

where $\hat{A}(\hat{x},\hat{t})$ is the amplitude of the surface disturbance and $c = (gh)^{1/2}$, with

$$\mu = \frac{3c}{2h}, \quad \hat{\Delta} = U - c \quad \text{and} \quad \lambda = \frac{ch^2}{6}. \tag{11}$$

This equation can be non-dimensionalized, using the velocity scale c and length scale h, to the form

$$-A^*_{t^*} - \Delta^* A^*_x + \frac{3}{2}A^* A^*_x + \frac{1}{6}A^*_{xxx} + \frac{1}{2}F^*_x = 0, \tag{12}$$

where $\Delta^* = \frac{U}{c} - 1$ is the variation from a Froude number of one, and then this can be rearranged to the canonical form

$$-A_t - \Delta A_x + 6AA_x + A_{xxx} + F_x = 0 \tag{13}$$

with scales $t = t^*/6$, $A = \frac{3}{2}A^*$, $F = \frac{9}{2}F^*$ and $\Delta = 6\Delta^*$. In linear hydraulic theory a Froude number of one corresponds to waves of infinite length, and so the weakly-nonlinear (KdV) theory is valid close to this value. Grimshaw et al [7. 8] did an analysis of this equation and then solved it numerically using finite differences for the case where

$$F(x) = \frac{F_M}{2}\big(\tanh\gamma x - \tanh\gamma(x-L)\big). \tag{14}$$

This form (14) of $F(x)$ represents an upward step of height F_M at $x = 0$ and a drop back to the original level at $x = L$ and the value of $1/\gamma$ represents the distance over which this elevation change occurs.

In the study group we decided to see if we could reproduce these solutions to verify that such solitary waves formed over a step. Following the numerical scheme outlined in Grimshaw et al [7] we computed solutions for a number of cases. Figure 3 shows a sequence of waves generated at different times for one such case. This figure shows a train of solitary waves and agrees well with the results shown in Grimshaw et al [7], thus verifying that our numerical method is working correctly. These waves are unfortunately

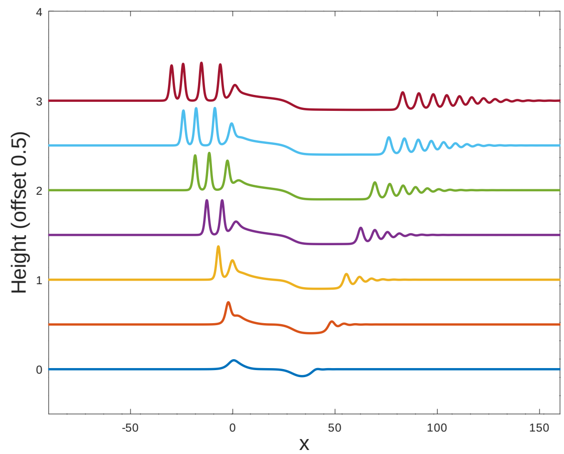

Figure 3: A time sequence of the solution to the KdV equation for flow over a step in the bottom topography. Time moves upward in intervals of 10 units with an offset of 0.5. Here $F_M = 0.1$, $\gamma = 0.25$ and $\Delta = 0.2$, that is, $Fr \approx 1.03$. Amplitude is reduced by a factor of 2.5 for clarity. The step rises at $x = 0$ and drops at $x = 30$.

occurring upstream of the step, which is an unlikely scenario for the chessboard waves as they appear to occur in the shallow water near to the shore.

Waves propagating to the left are of solitary form and occur as the flow hits the step, while those travelling to the right are normal ocean waves and occur after the drop off. This would suggest that one possible mechanism for formation is the surge of waves approaching a shelf in the bottom topography and shedding solitary waves in regular succession. The fact that conditions at Isle de Rhe consist of the meeting of two seas suggests that a likely mechanism for the formation is two trains of sinusoidal waves impacting a plateau in the topography from different directions. Whether the ultimate cause turns out to be a plateau or some effect of the meteorological crossing, it would seem this is the appropriate model. Further experiments with the parameter values would be necessary to determine in what circumstances the waves will occur.

4 Final comments

The chessboard waves observed at Isle de Rhe were considered and the conclusion is that these waves are an interaction of sequences of solitary waves shed by sea swell as it approaches the region from different directions. The regularity of the appearance of such waves suggests that the conditions under which they form occur regularly. It would therefore seem likely that it is a combination of weather systems with bottom topography that is the instigator of these waves, but that the primary factor is the local topography. A detailed examination of the weather conditions when such events occur and an accurate map of the local bottom topography, as well as more detailed simulations of the full Euler equations would be required to confirm or refute this suggestion.

References

[1] Airy GB. Tides and waves. Encyclopedia Metropolitana, 1845 article 192, pp. 241–396.

[2] Boussinesq J. Thorie de l'intumescence liquide, applele onde solitaire ou de translation, se propageant dans un canal rectangulaire. Comptes Rendus de l'Academie des Sciences. **72** (1871), 755–759.

[3] Dysthe KB. Note on a modification to the nonlinear Schrodinger equation for application to deep water waves. Proc. R. Soc. Lond. A, **369**, (1979) 105–114. doi:10.1098/rspa.1979.0154 .

[4] Duba CT, Please CE, Hocking GC, Born K and Kennealy M. Rogue Waves. Proceedings of the Mathematics in Industry Study Group, Cape Town, South Africa, 2019, pp 21–33.

[5] Forbes LK and Schwartz LW. Free-surface flow over a semicircular obstruction. J. Fluid Mech., **114**, (1982), 299–314. doi: 10.1017/S0022112082000160.

[6] Forbes LK. Non-linear, drag-free flow over a submerged semi-elliptical body. J. Eng. Math. **16**, (1982), 171–180. doi: 10.1007/BF00042552.

[7] Grimshaw RHJ, Zhang DH and Chow KW. Generation of solitary waves by transcritical flow over a step. J. Fluid Mech., **587**, (2007) 235-254. doi:10.1017/S0022112007007355.

[8] Grimshaw RHJ. Transcritical flow past an obstacle. ANZIAM J., **52**, (2010) 2–26. doi: 10.1017/s1446181111000599.

[9] Holmes RJ, Hocking GC, Forbes LK and Baillard NY. Euro. J. Appl. Maths. **24**(2013), 213–230. doi: 10.1017/S0956792512000381.

[10] Korteweg DJ and de Vries G. On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves. Philosophical Mag. 5th Series, **39**, (1895), 422–443. doi:10.1080/14786449508620739.

[11] Lamb H. 1932, Hydrodynamics, 6th ed., Cambridge University Press, Cambridge, United Kingdom, 1932.

[12] Stokes GG. On the theory of oscillatory waves. Trans. Cambridge Philos. Soc. **8**, (1847), 441–455. Reprinted in: Stokes, GG. Mathematical and Physical Papers, Volume I, 1880, Cambridge University Press, pp. 197–229.

[13] Zhang Y and Zhu S. Open channel flow past a bottom obstruction. J. Eng. Math., **30**, (1996), 487–499. doi: 10.1007/BF00049248.